# Calibration for Network Analysers and Parameter Conversions 

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## Contents

1 Introduction ..... 1
2 12-term calibration ..... 1
2.1 One-port calibration ..... 1
2.2 Two-port calibration ..... 3
3 8-term error model ..... 6
4 Network transformations ..... 6
4.1 Change of port impedance ..... 7
4.1.1 Same impedance for all ports ..... 7
4.1.2 Different port impedances ..... 7
4.2 General transformation of network matrices ..... 7
4.2.1 Example: from impedance parameters to chain parameters ..... 8
4.2.2 From impedance parameters to scattering parameters ..... 8

## 1 Introduction

To produce meaningful measurement results with a network analyser, it needs to be calibrated because the network analyser incorporates large systematic errors. To make reasonable measurements, these systematic errors need to be removed mathematically. The process of determining the error terms is called calibration. After the calibration has been performed and the error terms are known, they can be organised in a so-called error adapter. This is a fictitious network which is situated between the ideal network analyser and the device under test (DUT).

To measure the DUT's S-parameters, the network analyser's measurements are post-processed and the error adapter is mathematically removed.

## 2 12-term calibration

### 2.1 One-port calibration

For port 1 of the VNA, the model shown in Figure 1. A wave coming from the VNA's port 1 enters the error adapter at node $a_{0}$ and propagates through the error adapter. It is reflected by the DUT (blue nodes), propagates back through the error adapter and enters the VNA at node $b_{0}$.


Figure 1: Signal flow graph of a DUT connected to port 1 through the error adapter

The VNA cannot measure $S_{11}$, but only $S_{11}^{\prime}$ which is the ratio

$$
\begin{equation*}
S_{11}^{\prime}=\frac{b_{0}}{a_{0}} \tag{1}
\end{equation*}
$$

From the signal flow graph, one can find that

$$
\left.\begin{array}{l}
b_{1}=S_{11} a_{1}  \tag{2}\\
a_{1}=A a_{0}+B b_{1}
\end{array}\right\} b_{1}=a_{0} \frac{A S_{11}}{1-B S_{11}}
$$

The relationship

$$
\begin{equation*}
b_{0}=D a_{0}+C b_{1} \tag{3}
\end{equation*}
$$

can also be read directly from the signal flow graph, so together with Equation 2 one finds

$$
\begin{equation*}
b_{0}=D a_{0}+a_{0} \frac{A C S_{11}}{1-B S_{11}} \tag{4}
\end{equation*}
$$

Now one can find

$$
\begin{equation*}
S_{11}^{\prime}=\frac{b_{0}}{a_{0}}=D+\frac{A C S_{11}}{1-B S_{11}} \tag{5}
\end{equation*}
$$

It is also possible to solve Equation 5 for $S_{11}$ :

$$
\begin{equation*}
S_{11}=\frac{S_{11}^{\prime}-D}{B S_{11}+A C-B D} \tag{6}
\end{equation*}
$$

The expression $A C-B D$ can be abbreviated as $\Delta e_{1}$, and this simplifies Equation 6 to

$$
\begin{equation*}
S_{11}=\frac{S_{11}^{\prime}-D}{B S_{11}^{\prime}+\Delta e_{1}} \tag{7}
\end{equation*}
$$

Thus, if a load with reflection coefficient $S_{11}$ is connected to the error adapter, the $S_{11}^{\prime}$ according to Equation 5 is measured. With the aid of Equation 7, it becomes possible to find the actual value $S_{11}$, provided the error terms $B, D$ and $\Delta e_{1}$ are known.

However, before further calculations are done, one should notice a simplification. Consider the modified signal flow graph shown in Figure 2. When this graph is analysed, one finds that it has the same transfer function for $S_{11}^{\prime}$ as the flow graph used before, but it has the advantage that we need one error term less.


Figure 2: Modified one-port signal flow graph
Therefore, in the following steps, the modified signal flow graph is used.
For port 2 of the VNA, the signal flow graph shown in Figure 3 holds. Again, the DUT uses blue nodes. A wave coming from the VNA port 2 enters the signal flow graph from the right, propagates through the error adapter, gets reflected by the DUT's $S_{22}$ and after propagating back through the error adapter it enters the VNA at node $b_{3}$.


Figure 3: Signal flow graph for a one-port DUT connected to port 2

To find the error terms for port 1, Equation 5 is rearranged as follows

$$
\begin{equation*}
S_{11}^{\prime}=B S_{11}^{\prime} \Gamma+\Delta e \Gamma+D \tag{8}
\end{equation*}
$$

There are 3 error terms ( $B, D$ and $\Delta e_{1}$ ), so 3 independent measurements are required for each frequency point to calculate the actual reflection coefficient from the measured one. Therefore, at each frequency point, $\Gamma$ is known and $S_{11}^{\prime}$ can be measured. This leads to an equation system like

$$
\left(\begin{array}{ccc}
S_{11,1}^{\prime} S_{11,1} & 1 & S_{11,1}  \tag{9}\\
S_{11,2}^{\prime} S_{11,2} & 1 & S_{11,2} \\
S_{11,3}^{\prime} S_{11,3} & 1 & S_{11,3}
\end{array}\right) \cdot\left(\begin{array}{c}
B \\
D \\
\Delta e_{1}
\end{array}\right)=\left(\begin{array}{c}
S_{11,1}^{\prime} \\
S_{11,2}^{\prime} \\
S_{11,3}^{\prime}
\end{array}\right)
$$

which needs to be solved. For port 2, the procedure is exactly the same, except that different error terms are used as follows

$$
\left(\begin{array}{ccc}
S_{22,1}^{\prime} S_{22,1} & 1 & S_{22,1}  \tag{10}\\
S_{22,2}^{\prime} S_{22,2} & 1 & S_{22,2} \\
S_{22,3}^{\prime} S_{22,3} & 1 & S_{22,3}
\end{array}\right) \cdot\left(\begin{array}{c}
K \\
L \\
\Delta e_{2}
\end{array}\right)=\left(\begin{array}{c}
S_{22,1}^{\prime} \\
S_{22,2}^{\prime} \\
S_{22,3}^{\prime}
\end{array}\right)
$$

and

$$
\begin{equation*}
\Delta e_{2}=H I-K L \tag{11}
\end{equation*}
$$

The actual $S_{22}$ is found via

$$
\begin{equation*}
S_{22}=\frac{S_{22}^{\prime}-L}{K S_{22}^{\prime}+\Delta e_{2}} \tag{12}
\end{equation*}
$$

### 2.2 Two-port calibration

The DUT is now not a one-port but a two-port device with the four S-parameters $S_{11}, S_{21}, S_{12}$ and $S_{22}$. The signal flow graph for the forward direction is shown in Figure 4.


Figure 4: Signal flow graph for two port, forward direction
The return loss is found with

$$
\begin{equation*}
S_{11}^{\prime}=\frac{b_{0}}{a_{0}} \tag{13}
\end{equation*}
$$

and the forward transfer is

$$
\begin{equation*}
S_{21}^{\prime}=\frac{b_{3}}{a_{0}} \tag{14}
\end{equation*}
$$

We now develop the equations to express $S_{11}^{\prime}$ and $S_{21}^{\prime}$ in terms of the DUT's S-parameters and the error terms $A C, B, D, E, F$ and $G$. This procedure is quite lengthy, but in the end, one finds

$$
\begin{equation*}
S_{11}^{\prime}=\frac{b_{0}}{a_{0}}=D+\frac{A C\left(S_{11}+E\left(S_{12} S_{21}-S_{11} S_{22}\right)\right)}{B E\left(S_{11} S_{22}-S_{12} S_{21}\right)-E S_{22}-B S_{11}+1} \tag{15}
\end{equation*}
$$

and

$$
\begin{equation*}
S_{21}^{\prime}=\frac{b_{3}}{a_{0}}=G+\frac{F S_{21}}{B E\left(S_{11} S_{22}-S_{12} S_{21}\right)-E S_{22}-B S_{11}+1} \tag{16}
\end{equation*}
$$

The signal flow graph for the reverse direction is shown in Figure 5.


Figure 5: Signal flor graph for two port, reverse direction
Since it is basically the mirror image of Figure 4, the S-parameters can be found by interchaning the variable names. Then, one finds

$$
\begin{equation*}
S_{22}^{\prime}=\frac{b_{3}}{a_{3}}=L+\frac{H I\left(S_{22}+M\left(S_{12} S_{21}-S_{11} S_{22}\right)\right)}{K M\left(S_{11} S_{22}-S_{12} S_{21}\right)-M S_{11}-K S_{22}+1} \tag{17}
\end{equation*}
$$

and

$$
\begin{equation*}
S_{12}^{\prime}=\frac{b_{0}}{a_{3}}=P+\frac{N S_{12}}{K M\left(S_{11} S_{22}-S_{12} S_{21}\right)-M S_{11}-K S_{22}+1} . \tag{18}
\end{equation*}
$$

We now have an equation system for $S_{11}^{\prime}, S_{21}^{\prime}, S_{12}^{\prime}$ and $S_{22}^{\prime}$ which needs to be solved for $S_{11}, S_{21}$, $S_{12}$ and $S_{22}$. The procedure is quite cumbersome. However, one can find

$$
\begin{align*}
S_{11}= & \frac{\frac{S_{11}^{\prime}-D}{A C}\left(1+K \frac{S_{22}^{\prime}-L}{H I}\right)-E \frac{S_{21}^{\prime}-G}{F} \frac{S_{12}^{\prime}-P}{N}}{n}  \tag{19}\\
S_{21}= & \frac{\frac{S_{21}^{\prime}-G}{F}\left(1+(K-E) \frac{S_{22}^{\prime}-L}{H I}\right)}{n}  \tag{20}\\
S_{12}= & \frac{\frac{S_{12}^{\prime}-P}{N}\left(1+(B-M) \frac{S_{11}^{\prime}-D}{A C}\right)}{n}  \tag{21}\\
S_{22}= & \frac{\frac{S_{22}^{\prime}-L}{H I}\left(1+B \frac{S_{11}^{\prime}-D}{A C}\right)-M \frac{S_{21}^{\prime}-G}{F} \frac{S_{12}^{\prime}-P}{N}}{n} \tag{22}
\end{align*}
$$

with the abbreviation

$$
\begin{equation*}
n=\left(1+B \frac{S_{11}^{\prime}-D}{A C}\right)\left(1+K \frac{S_{22}^{\prime}-L}{H I}\right)-E M \frac{S_{21}^{\prime}-G}{F} \frac{S_{12}^{\prime}-P}{N} \tag{23}
\end{equation*}
$$

To find all error terms, the error terms for port 1 are calculated by means of Equation 9. For port 2, the error terms are determined with Equation 10. Then, the error terms $G$ and $P$ can be determined directly if both ports are terminated and $S_{21}^{\prime}$ as well as $S_{12}^{\prime}$ are measured. Then,

$$
\begin{equation*}
G=\frac{b_{3}}{a_{0}}=S_{21}^{\prime} \tag{24}
\end{equation*}
$$

and

$$
\begin{equation*}
P=\frac{b_{0}}{a_{3}}=S_{12}^{\prime} \tag{25}
\end{equation*}
$$



Figure 6: Measuring port 2 match

The error terms for the two-port calibration can be found as follows. Both ports are connected together directly. In this case, the signal flow graph in Figure 6 applies.
With a measurement of

$$
\begin{equation*}
S_{11}^{\prime}=\frac{b_{0}}{a_{0}} \tag{26}
\end{equation*}
$$

it becomes possible to determine the error term $E$, which takes a mismatch at port 2 of the VNA into account. With a measurement of

$$
\begin{equation*}
S_{21}^{\prime}=\frac{b_{3}}{a_{0}} \tag{27}
\end{equation*}
$$

the error term $F$ can be determined.
The error term $E$ can be determined from $S_{11}^{\prime}$ because it has, in this configuration, the same role as $S_{11}$ of a one-port DUT. Since port 1 was calibrated before, it becomes possible to determine $E$ by means of

$$
\begin{equation*}
E=\frac{S_{11}^{\prime}-D}{B S_{11}^{\prime}+\Delta e_{1}} \tag{28}
\end{equation*}
$$

For the node $x$, the equation

$$
\begin{equation*}
x=a_{0}+B E x \tag{29}
\end{equation*}
$$

holds, which can be solved for $x$ :

$$
\begin{equation*}
x=a_{0} \frac{1}{1-B E} \tag{30}
\end{equation*}
$$

Now, $b_{3}$ can be written as

$$
\begin{equation*}
b_{3}=F x+G a_{0}=a_{0}\left(\frac{F}{1-B E}+G\right) \tag{31}
\end{equation*}
$$

and hence

$$
\begin{equation*}
S_{21}^{\prime}=\frac{F}{1-B E}+G \tag{32}
\end{equation*}
$$

If solved for $F$, this yields

$$
\begin{equation*}
F=\left(S_{21}^{\prime}-G\right) \cdot(1-B E) \tag{33}
\end{equation*}
$$

which allows to determine the error term $F$, provided $G, B$ and $E$ are known.
On the other hand, if $S_{12}^{\prime}$ and $S_{22}^{\prime}$ are measured, the signal flow graph shown in Figure 7 needs to be taken into account.
With a measurement of

$$
\begin{equation*}
S_{22}^{\prime}=\frac{b_{3}}{a_{3}} \tag{34}
\end{equation*}
$$

it becomes possible to determine $M$, and measuring

$$
\begin{equation*}
S_{12}^{\prime}=\frac{b_{0}}{a_{3}} \tag{35}
\end{equation*}
$$

allows to determine $N$. The individual steps in the calculation are basically the same as before and one will find

$$
\begin{equation*}
M=\frac{S_{22}^{\prime}-L}{K S_{22}^{\prime}+\Delta e_{2}} \tag{36}
\end{equation*}
$$

and

$$
\begin{equation*}
N=\left(S_{12}^{\prime}-P\right) \cdot(1-K M) \tag{37}
\end{equation*}
$$



Figure 7: Measuring port 1 match

## 3 Network transformations

See [2, 3].

### 3.1 Change of port impedance

The use case for this is for instance when a $75 \Omega$ device is to be measured with a $50 \Omega$ network analyser, or when a device has ports with different impedances, i.e. port 1 has $50 \Omega$ and port 2 has $75 \Omega$ for instance. Then we can calculate the scattering parameters the device would have if all ports were matched.

### 3.1.1 Same impedance for all ports

With the following abbreviation

$$
\begin{equation*}
r=\frac{Z_{0}^{\prime}-Z_{0}}{Z_{0}^{\prime}+Z_{0}} \tag{38}
\end{equation*}
$$

the rescaling of the scattering parameters to a new port impedance can be done with

$$
\begin{equation*}
\mathbf{S}^{\prime}=(\mathbf{I}-r \mathbf{S})^{-1} \cdot(\mathbf{S}-r \mathbf{I}) \tag{39}
\end{equation*}
$$

### 3.1.2 Different port impedances

The easiest way is via the impedance matrix

$$
\begin{equation*}
\mathbf{Z}=(\mathbf{I}-\mathbf{S})^{-1} \cdot(\mathbf{I}+\mathbf{S}) \cdot \mathbf{Z}_{0} \tag{40}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathbf{Z}_{0}=\operatorname{diag}\left(Z_{1}, Z_{2}, \ldots, Z_{n}\right) \tag{41}
\end{equation*}
$$

and $Z_{i}$ being the port impedance of port $i$. Then,

$$
\begin{equation*}
\mathbf{S}^{\prime}=\left(\mathbf{Z}-\mathbf{Z}_{0}^{\prime}\right)\left(\mathbf{Z}+\mathbf{Z}_{0}^{\prime}\right)^{-1} \tag{42}
\end{equation*}
$$

and obviously

$$
\begin{equation*}
\mathbf{Z}_{0}^{\prime}=\operatorname{diag}\left(Z_{1}^{\prime}, Z_{2}^{\prime}, \ldots, Z_{n}^{\prime}\right) \tag{43}
\end{equation*}
$$

So we have

$$
\begin{equation*}
\mathbf{S}^{\prime}=\left((\mathbf{I}-\mathbf{S})^{-1}(\mathbf{I}+\mathbf{S}) \mathbf{Z}_{0}-\mathbf{Z}_{0}^{\prime}\right)\left((\mathbf{I}-\mathbf{S})^{-1}(\mathbf{I}+\mathbf{S}) \mathbf{Z}_{0}+\mathbf{Z}_{0}^{\prime}\right)^{-1} \tag{44}
\end{equation*}
$$

Care must be taken when this is implemented in Matlab. Due to the numeric algorithms used, there are different operators for the left-inverse multiplication and the right-inverse multiplication. For instance, the above would better be written as

```
Ss = ((I-S) \ (I+S) * ZO - ZOs) / ((I-S) \ (I+S) * ZO + ZOs)
```

instead of
$\mathrm{Ss}=(\operatorname{inv}(\mathrm{I}-\mathrm{S}) *(\mathrm{I}+\mathrm{S}) * \mathrm{ZO}-\mathrm{ZOs}) * \operatorname{inv}(\operatorname{inv}(\mathrm{I}-\mathrm{S}) *(\mathrm{I}+\mathrm{S}) * \mathrm{ZO}+\mathrm{ZOs})$
because the latter may lead to some issues due to rounding etc.

### 3.2 General transformation of network matrices

Need for these transformations arises for instance when one needs to calculate the scattering parameters of a cascade of networks for which only the impedance parameters are given.
Generally, a network representation relates certain input signals $\mathbf{M}$ to some output signals $\mathbf{N}$ as follows:

$$
\begin{equation*}
\mathbf{N}=\mathbf{R} \mathbf{M} \tag{45}
\end{equation*}
$$

We want to transform this representation to a different one, e.g. we want to convert scattering parameters to chain parameters. The new representation may have the general form

$$
\begin{equation*}
\mathbf{N}^{\prime}=\mathbf{R}^{\prime} \mathbf{M}^{\prime} \tag{46}
\end{equation*}
$$

from which $\mathbf{N}^{\prime}$ and $\mathbf{M}^{\prime}$ are known and $\mathbf{R}$ shall be found.
To find the new matrix $\mathbf{R}^{\prime}$, the relations between $\mathbf{M}, \mathbf{M}^{\prime}, \mathbf{N}$ and $\mathbf{N}^{\prime}$ need to be taken into account. We have

$$
\binom{\mathbf{M}^{\prime}}{\mathbf{N}^{\prime}}=\left(\begin{array}{ll}
\mathbf{X}_{11} & \mathbf{X}_{12}  \tag{47}\\
\mathbf{X}_{21} & \mathbf{X}_{22}
\end{array}\right)\binom{\mathbf{M}}{\mathbf{N}}
$$

and therefore

$$
\begin{align*}
\mathbf{M}^{\prime} & =\mathbf{X}_{11} \mathbf{M}+\mathbf{X}_{12} \mathbf{N} \\
\mathbf{N}^{\prime} & =\mathbf{X}_{21} \mathbf{M}+\mathbf{X}_{22} \mathbf{N} \tag{48}
\end{align*}
$$

and we want to write $\mathbf{N}^{\prime}$ in terms of $\mathbf{M}^{\prime}$. Since $\mathbf{N}=\mathbf{R} \mathbf{M}$, we find

$$
\begin{align*}
\mathbf{M}^{\prime} & =\left(\mathbf{X}_{11}+\mathbf{X}_{12} \mathbf{R}\right) \mathbf{M} \\
\mathbf{N}^{\prime} & =\left(\mathbf{X}_{21}+\mathbf{X}_{22} \mathbf{R}\right) \mathbf{M} \tag{49}
\end{align*}
$$

from which we can also determine

$$
\begin{equation*}
\mathbf{M}=\left(\mathbf{X}_{11}+\mathbf{X}_{12} \mathbf{R}\right)^{-1} \mathbf{M}^{\prime} \tag{50}
\end{equation*}
$$

This yields

$$
\begin{equation*}
\mathbf{N}^{\prime}=\left(\mathbf{X}_{21}+\mathbf{X}_{22} \mathbf{R}\right)\left(\mathbf{X}_{11}+\mathbf{X}_{12} \mathbf{R}\right)^{-1} \mathbf{M}^{\prime} \tag{51}
\end{equation*}
$$

from which we can read off $\mathbf{R}^{\prime}$ directly as

$$
\begin{equation*}
\mathbf{R}^{\prime}=\left(\mathbf{X}_{21}+\mathbf{X}_{22} \mathbf{R}\right)\left(\mathbf{X}_{11}+\mathbf{X}_{12} \mathbf{R}\right)^{-1} \tag{52}
\end{equation*}
$$

### 3.2.1 Example: from impedance parameters to chain parameters

We want to convert from impedance parameters

$$
\begin{equation*}
\binom{V_{1}}{V_{2}}=\mathbf{Z}\binom{I_{1}}{I_{2}} \tag{53}
\end{equation*}
$$

to chain parameters (ABCD parameters):

$$
\begin{equation*}
\binom{V_{1}}{I_{1}}=\mathbf{C}\binom{V_{2}}{-I_{2}} \tag{54}
\end{equation*}
$$

Obviously, the input variables in case of the impedance parameters are $I_{1}$ and $I_{2}$ and $V_{2}$ and $-I_{2}$ for the chain parameters. On the other hand, we have $V_{1}$ and $V_{2}$ as output variables for the impedance parameters and $V_{1}$ and $I_{1}$ for the chain parameters. This may be written as

$$
\left(\begin{array}{c}
V_{2}  \tag{55}\\
-I_{2} \\
V_{1} \\
I_{1}
\end{array}\right)=\left(\begin{array}{cc|cc}
0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 \\
\hline 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{array}\right)\left(\begin{array}{c}
I_{1} \\
I_{2} \\
V_{1} \\
V_{2}
\end{array}\right)
$$

from which we find $\mathbf{X}$ as follows:

$$
\mathbf{X}_{11}=\left(\begin{array}{cc}
0 & 0  \tag{56}\\
0 & -1
\end{array}\right) \quad, \quad \mathbf{X}_{12}=\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right) \quad, \quad \mathbf{X}_{21}=\left(\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right) \quad, \quad \mathbf{X}_{22}=\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right)
$$

Thus, the conversion from impedance to ABCD parameters may be performed with

$$
\mathbf{C}=\left(\left(\begin{array}{ll}
0 & 0  \tag{57}\\
1 & 0
\end{array}\right)+\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) \mathbf{Z}\right)\left(\left(\begin{array}{cc}
0 & 0 \\
0 & -1
\end{array}\right)+\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right) \mathbf{Z}\right)^{-1}
$$

### 3.2.2 From impedance parameters to scattering parameters

Voltage waves:

$$
\begin{align*}
a & =\frac{1}{2}\left(V+Z_{0} I\right)  \tag{58}\\
b & =\frac{1}{2}\left(V-Z_{0} I\right)
\end{align*}
$$

From here, we find the transformation

$$
\left(\begin{array}{l}
a_{1}  \tag{59}\\
a_{2} \\
b_{1} \\
b_{2}
\end{array}\right)=\frac{1}{2}\left(\begin{array}{cccc}
Z_{0} & 0 & 1 & 0 \\
0 & Z_{0} & 0 & 1 \\
-Z_{0} & 0 & 1 & 0 \\
0 & -Z_{0} & 0 & 1
\end{array}\right)\left(\begin{array}{l}
I_{1} \\
I_{2} \\
V_{1} \\
V_{2}
\end{array}\right)
$$

which may be expanded for an arbitrary number of ports, i.e. it is not restricted to 2-port networks. It is even possible to have a different characteristic impedance for each port.

Equation 66 leads to the general conversion

$$
\begin{equation*}
\mathbf{S}=\left(\mathbf{Z}-\operatorname{diag}\left(Z_{1}, Z_{2}, \ldots, Z_{n}\right)\right)\left(\mathbf{Z}+\operatorname{diag}\left(Z_{1}, Z_{2}, \ldots, Z_{n}\right)\right)^{-1} \tag{60}
\end{equation*}
$$

where the $Z_{i}$ are the characteristic impedances of the ports. Note that this is the exact vector equivalent of

$$
\begin{equation*}
\Gamma=\frac{Z-Z_{0}}{Z+Z_{0}} \tag{61}
\end{equation*}
$$

which is used to derive the reflection coefficient of a 1-port device.
The inverse conversion is easily found with

$$
\begin{equation*}
V=a+b \tag{62}
\end{equation*}
$$

and

$$
\begin{equation*}
I=Y_{0}(a-b) \tag{63}
\end{equation*}
$$

which leads to:

$$
\left(\begin{array}{c}
I_{1}  \tag{64}\\
I_{2} \\
V_{1} \\
V_{2}
\end{array}\right)=\left(\begin{array}{cccc}
Y_{0} & 0 & -Y_{0} & 0 \\
0 & Y_{0} & 0 & -Y_{0} \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{array}\right)\left(\begin{array}{l}
a_{1} \\
a_{2} \\
b_{1} \\
b_{2}
\end{array}\right)
$$

From here, one finds

$$
\begin{equation*}
\mathbf{Z}=(\mathbf{I}+\mathbf{S})\left(\operatorname{diag}\left(Y_{1}, Y_{2}, \ldots, Y_{n}\right)-\operatorname{diag}\left(Y_{1}, Y_{2}, \ldots, Y_{n}\right) \mathbf{S}\right)^{-1} \tag{65}
\end{equation*}
$$

but this can be written as

$$
\begin{array}{r}
\mathbf{Z}=(\mathbf{I}+\mathbf{S})\left(\operatorname{diag}\left(Y_{1}, Y_{2}, \ldots, Y_{n}\right) \cdot(\mathbf{I}-\mathbf{S})\right)^{-1} \\
=(\mathbf{I}+\mathbf{S})(\mathbf{I}-\mathbf{S})^{-1} \cdot \mathbf{Z}_{0} . \tag{66}
\end{array}
$$

This is because

$$
\begin{equation*}
(\mathbf{A} \cdot \mathbf{B})^{-1}=\mathbf{B}^{-1} \cdot \mathbf{A}^{-1} \tag{67}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{diag}\left(Y_{1}, Y_{2}, \ldots, Y_{n}\right)^{-1}=\operatorname{diag}\left(Z_{1}, Z_{2}, \ldots, Z_{n}\right) \tag{68}
\end{equation*}
$$

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