Multi-hole Chebyshev Directional Couplers

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Abstract

Short theory about multi-hole directional couplers in waveguide technology, simulation of couplers using Matlab and full-wave simulation and construction and testing of a Ka-band coupler.

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1 Introduction

For the design of a waveguide directional coupler, the Chebyshev polynomials are used (however, there is also such a thing as a binomial directional coupler, but their frequency response is not as flat as for the Chebyshev ones). Definition of the n-th Chebyshev polynomial:

$$T_n(x) = 2\,x\,T_{n-1}(x) - T_{n-2}(x) \tag{1}$$

The first two Chebyshev polynomials are

$$T_0(x) = 1 \tag{2}$$

and

$$T_1(x) = x \tag{3}$$

and with the aid of Equation 1 the polynomials in Table 1 can be found.

Table 1: The first 6 Chebyshev polynomials

n	$T_n(x)$
0	1
1	x
2	$2\,x^2 - 1$
3	$4\,x^3 - 3\,x$
4	$8\,x^4 - 8\,x^2 + 1$
5	$16\,x^5 - 20\,x^3 + 5\,x$
6	$32\,x^6 - 48\,x^4 + 18\,x^2 - 1$

In standard Matlab (or Octave), there is no function available to calculate the coefficients of the Chebyshev polynomials. The code shown in Listing 1 (see Appendix A) is a simple recursive implementation of an algorithm with returns the `n`-th Chebyshev polynomial, using Equation 1 to Equation 3.

Important properties of Chebyshev polynomials:

- The Chebyshev polynomials are defined for x-values between -1 and $+1$.
- The n-th Chebyshev polynomial has exactly n zeros in the interval $[-1, 1]$.
- They have the so-called equal-ripple property: this means the polynomials oscillate, but their maximum and minimum values do not exceed ± 1.

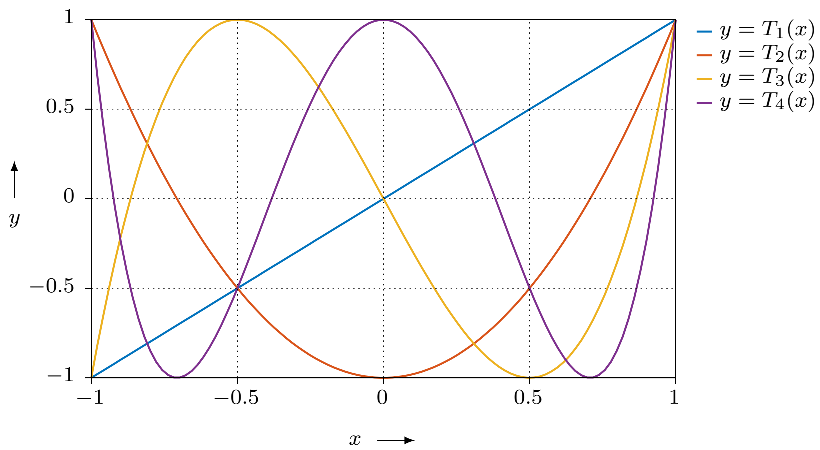


Figure 1: Plot of some Chebyshev polynomials

1.1 Effect of holes in waveguide walls

Consider two waveguides with a common broad wall. If there is one single hole in this wall, a portion of a wave propagating in one of the waveguides "leaks" into the other waveguide. This can be expressed by means of the voltage coupling coefficient

$$U = \mathrm{j}\,\frac{2\,\pi}{a\,b\,\lambda_g}\left(M_x\,H_x^{(1)}\,H_x^{(2)} + M_z\,H_z^{(1)}\,H_z^{(2)} - P_y\,E_y^{(1)}\,E_y^{(2)}\right) \tag{4}$$

where:

- M_x and M_z are the x and z component of the magnetic polarisability of the hole,
- P_y is the electric polarisability,
- H_x and H_z are the magnetic field components in waveguide 1 or 2,
- E_y are the electric field components in waveguide 1 or 2.

The waveguide dimensions are a and b, where a is the size of the broad wall, and λ_g is the guided wavelength.

The polarisability of a hole having a circular shape can be derived, but we skip this derivation here and give just the expressions

$$M_x = M_z = \frac{d^3}{6} \quad , \quad P_y = \frac{d^3}{12} \tag{5}$$

where d is the hole's diameter. Other hole shapes than circular can be used; e.g. there are directional couplers using elliptical holes or even holes in the shape of a cross. We use the round holes because they are the simplest to make.

From waveguide theory, one can find the values of the magnetic field components by

$$H_x = -\sin\frac{\pi\,x}{a}\cdot \mathrm{e}^{-\mathrm{j}\,\gamma\,z} \tag{6}$$

$$H_z = \mathrm{j}\frac{\lambda_g}{2\,a}\cos\frac{\pi\,x}{a}\cdot \mathrm{e}^{-\mathrm{j}\,\gamma\,z} \tag{7}$$

$$E_y = \frac{\lambda_g}{\lambda}\sin\frac{\pi\,x}{a}\cdot \mathrm{e}^{-\mathrm{j}\,\gamma\,z} \tag{8}$$

we can substitute these expressions into Equation 4 and end up with Equation 9.

$$U = \mathrm{j}\,\frac{2\,\pi\,d^3}{a\,b\,\lambda_g}\left(\frac{1}{6}\sin^2\frac{\pi\,x}{a} + \frac{\lambda_g^2}{24\,a^2}\cos^2\frac{\pi\,x}{a} - \frac{\lambda_g^2}{12\,\lambda^2}\sin^2\frac{\pi\,x}{a}\right) \tag{9}$$

The variable x corresponds to the distance of the hole from the wall (Figure 2).

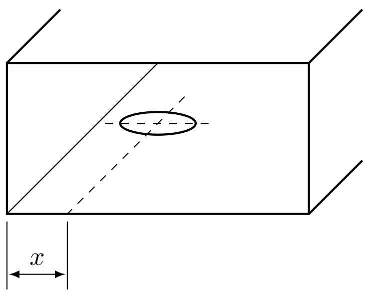


Figure 2: Waveguide with a single hole

1.2 Corrections

The voltage coupling coefficient given by Equation 9 is only valid if the holes are very small compared to the wavelength, and if the metal thickness is infinitely small. There are correction factors to take into account that the hole diameters are in the same order of magnitude as the wavelength and also to correct for thick metal walls.

1.2.1 Hole diameter correction

In practice, the size of the holes cannot be neglected. Assume round holes are used. Then they can be considered as tiny circular waveguides, having a certain cutoff frequency and attenuation. The correction factor for the hole size is

$$K_1 = \frac{2\,\lambda}{\pi\,\lambda_0}\tan\frac{\pi\,\lambda_0}{2\,\lambda} \tag{10}$$

where λ_0 is the resonant wavelength of the hole. From the theory of circular waveguides, one can find

$$\lambda_0 = 1.705\, d \tag{11}$$

for magnetic coupling, and

$$\lambda_0 = 1.305\, d \tag{12}$$

for electric coupling.

1.2.2 Wall thickness correction

As said previously, the holes can be viewed as tiny circular waveguides. Because they are so small, they are mostly operated below their cutoff frequency and thus the waves propagating through the holes are evanescent. The evanescent waves are attenuated depending on the metal thickness. The attenuation is described with

$$K_2 = \exp\left(-\frac{2\,\pi\,A}{\lambda_0}\sqrt{1-\frac{\lambda_0^2}{\lambda^2}}\right) \tag{13}$$

where A describes the attenuation of magnetic and electric field components depending on the metal thickness. For the magnetic field components,

$$A = 1.0064\, t + 0.04095\, d \tag{14}$$

and

$$A = 1.0103\, t + 0.02895\, d \tag{15}$$

for the electric field components.

1.3 Voltage coupling coefficient of a single hole

Using the previous definitions, the voltage coupling coefficient for one hole can be written as

$$U = \mathrm{j}\,\frac{2\,\pi\,d^3}{a\,b\,\lambda_g}\cdot(u_1+u_2+u_3) \tag{16}$$

where u_1 and u_2 are the magnetic coupling terms, and u_3 is the electric coupling term as follows:

$$u_1 = \frac{1}{6}\sin^2\frac{\pi\,x}{a}\cdot K_1\cdot K_2 \tag{17}$$

$$u_2 = \frac{\lambda_g^2}{24\,a^2}\cos^2\frac{\pi\,x}{a}\cdot K_1\cdot K_2 \tag{18}$$

$$u_3 = -\frac{\lambda_g^2}{12\,\lambda^2}\sin^2\frac{\pi\,x}{a}\cdot K_1\cdot K_2 \tag{19}$$

Care must be taken that, for K_1 and K_2, the corresponding terms for the magnetic or electric field, respectively, are taken into account.

So, with this expression for U, it is possible to design a single-hole directional coupler, like a Bethe-Hole coupler. While such a coupler is simple to design and can offer quite good directivity, it is inherently narrowband. If a directional coupler having a good directivity over a wide bandwidth is to be designed, a multi-hole coupler is the solution.

If two holes next to each other are used (Figure 3), the coupling coefficients of the two holes can be simply added.

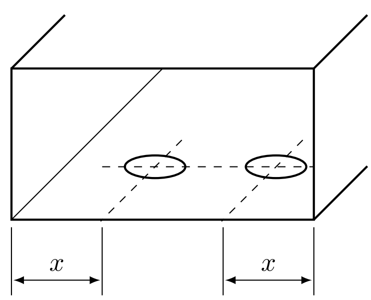


Figure 3: A coupler with two holes next to each other

2 Application of the Chebyshev polynomials

2.1 A first example

This is actually an example taken from Levy's paper, but slightly modified. Consider the directional coupler shown in Figure 4, having four holes and symmetrical voltage coupling coefficients (a_1, a_2, a_2, a_1).

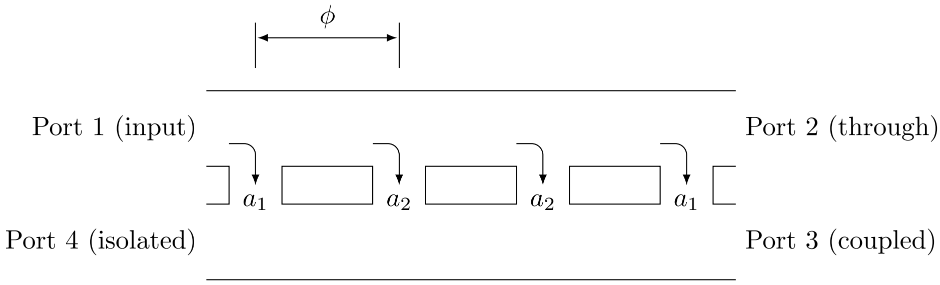


Figure 4: A directional coupler with four holes

The electrical distance, ϕ, between the holes is

$$\phi = \frac{2\,\pi\,\ell}{\lambda_g} \tag{20}$$

where ℓ is the physical distance between the holes.

Assume all ports are matched and that a wave of amplitude 1 enters on the input port. Since the coupler is not perfect, a portion of the incident wave will reach the isolated port; the net voltage reaching the isolated port is then

$$I = a_1 + a_2\,\mathrm{e}^{-2\,\mathrm{j}\,\phi} + a_2\,\mathrm{e}^{-4\,\mathrm{j}\,\phi} + a_1\,\mathrm{e}^{-6\,\mathrm{j}\,\phi} \tag{21}$$

where I stands for isolation. The coupling would be determined with

$$C = a_1\,\mathrm{e}^{-3\,\mathrm{j}\,\phi} + \mathrm{e}^{-\mathrm{j}\,\phi} a_2 \mathrm{e}^{-2\,\mathrm{j}\,\phi} + \mathrm{e}^{-2\,\mathrm{j}\,\phi} a_2 \mathrm{e}^{-\mathrm{j}\,\phi} + \mathrm{e}^{-3\,\mathrm{j}\,\phi} a_1 = (2\,a_1 + 2\,a_2)\,\mathrm{e}^{-3\,\mathrm{j}\,\phi} \tag{22}$$

and

$$|C| = 2\,a_1 + 2\,a_2 \quad . \tag{23}$$

From Equation 21, one can factor out $\mathrm{e}^{3\,\mathrm{j}\,\phi}$ as follows:

$$I = \mathrm{e}^{-3\,\mathrm{j}\,\phi}\left(a_1\,\mathrm{e}^{\mathrm{j}\,3\,\phi} + a_2\,\mathrm{e}^{\mathrm{j}\,\phi} + a_2\,\mathrm{e}^{-\mathrm{j}\,\phi} + a_1\,\mathrm{e}^{-3\,\mathrm{j}\,\phi}\right) \tag{24}$$

From Euler's identity, we know that

$$\mathrm{e}^{\mathrm{j}\,\phi} = \cos\phi + \mathrm{j}\sin\phi \tag{25}$$

and

$$\mathrm{e}^{-\mathrm{j}\,\phi} = \cos\phi - \mathrm{j}\sin\phi \quad , \tag{26}$$

so we can write

$$\mathrm{e}^{\mathrm{j}\,\phi} + \mathrm{e}^{-\mathrm{j}\phi} = 2\cos\phi \quad . \tag{27}$$

Therefore, Equation 24 can be written as

$$I = \mathrm{e}^{-3\,\mathrm{j}\,\phi}\,(2\,a_1\cos 3\,\phi + 2\,a_2\cos\phi) \quad . \tag{28}$$

In practice, we are not interested in the phase of I but only in the magnitude. For a directional coupler, we want I to be minimal to have maximum isolation.

Equation 28 can be written as the sum of two Chebyshev polynomials with the abbreviation $x = \cos\phi$ as follows[1]:

$$|I| = |2\,a_1\,T_3(x) + 2\,a_2\,T_1(x)| \tag{29}$$

[1] note that $\left|\mathrm{e}^{3\,\mathrm{j}\,\phi}\right| = 1$, so we can omit this term

To show this, we insert the actual Chebyshev polynomials and $x = \cos\phi$ into Equation 29:

$$|I| = \left|2\,a_1\left(4\cos^3\phi - 3\cos\phi\right) + 2\,a_2\cos\phi\right| \tag{30}$$

To further analyse this expression, we may use the identity

$$\cos^3\alpha = \frac{1}{4}\left(3\cos\alpha + \cos 3\,\alpha\right) \tag{31}$$

which yields

$$|I| = \left|2\,a_1\left(4\cdot\frac{1}{4}\left(3\cos\phi + \cos 3\,\phi\right) - 3\cos\phi\right) + 2\,a_2\cos\phi\right| \tag{32}$$

when inserted into Equation 30. This can be further simplified:

$$\begin{aligned}|I| &= \left|2\,a_1\left(\cancel{4}\cdot\frac{1}{\cancel{4}}\left(\cancel{3\cos\phi} + \cos 3\,\phi\right) - \cancel{3\cos\phi}\right) + 2\,a_2\cos\phi\right| \\ &= \left|2\,a_1\cos 3\,\phi + 2\,a_2\cos\phi\right|\end{aligned} \tag{33}$$

which has the same magnitude as Equation 28.

However, from Equation 29, we see that $|I|$ is essentially the sum of two Chebyshev polynomials, T_3 and T_1. To have the largest possible bandwidth while still having the flattest frequency response, $|I|$ should be equal to a Chebyshev polynomial,

$$|I| \propto |T_3(x)| \tag{34}$$

and we now want to find out how we shall choose the voltage coupling coefficients a_1 and a_2 such that Equation 34 holds.

The equal ripple property of the Chebyshev polynomials is only valid in the range $x \in [-1, 1]$. Therefore, we introduce a scaling factor, t:

$$|I| = |T_3(t\,x)| \tag{35}$$

The scaling factor must be determined such that $t\,x = \pm 1$ at the two corner frequencies of the directional coupler.

$$t\cos\phi = \pm 1 \quad \Leftrightarrow \quad t = \frac{1}{\cos\phi} \tag{36}$$

From Equation 29, we find

$$\begin{aligned}|I| &= |2\,a_1\,T_3(x) + 2\,a_2\,T_1(x)| \\ &= \left|2\,a_1\,\left(4\,x^3 - 3\,x\right) + 2\,a_2\,x\right| \\ &= \left|8\,a_1\,x^3 - 6\,a_1\,x + 2\,a_2\,x\right| \\ &= \left|8\,a_1\,x^3 - \left(6\,a_1 - 2\,a_2\right)x\right|\end{aligned} \tag{37}$$

and by applying the condition Equation 35, we find:

$$\left|8\,a_1\,x^3 - \left(6\,a_1 - 2\,a_2\right)x\right| = \left|4\,t^3\,x^3 - 3\,t\,x\right| \tag{38}$$

Since we are only concerned about the magnitude, we can do a comparison of coefficients:

$$8\,a_1 = 4\,t^3 \tag{39}$$

$$6\,a_1 - 2\,a_2 = 3\,t \tag{40}$$

Both, a_1 and a_2, can be expressed as follows:

$$a_1 = a_{11}\,t^3 + a_{12}\,t \tag{41}$$

$$a_2 = a_{21}\,t^3 + a_{22}\,t \tag{42}$$

We may write this equation system in matrix form, which allows to solve it easily using numeric software.

$$\begin{pmatrix} 8 & 0 \\ 6 & -2 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \tag{43}$$

The solution is

$$\begin{pmatrix} \frac{1}{2} & 0 \\ \frac{3}{2} & -\frac{3}{2} \end{pmatrix}$$

which is the same as the following:

$$a_1 = \frac{1}{2}\, t^3 \tag{44}$$

$$a_2 = \frac{3}{2}\, t^3 - \frac{3}{2}\, t \tag{45}$$

Now we can normalise a_2 with respect to a_1, as follows:

$$\frac{a_2}{a_1} = \frac{\frac{3}{2}\, t^3 - \frac{3}{2}\, t}{\frac{1}{2}\, t^3} = 3\frac{t^3 - t}{t^3} = 3\left(1 - \frac{1}{t^2}\right) = 3 - \frac{3}{t^2} \tag{46}$$

How shall one interpret this result? it means the following: if the voltage coupling coefficient a_2 is $\left(3 - \frac{3}{t^2}\right)$ times a_1, the directional coupler constructed in this way has a Chebyshev response and thus maximises bandwidth and directivity. How the actual calculations are done is shown in an example.

The polynomials of the kind in Equation 46 are tabulated by Young and Levy [2, p. 142]. Refer also to Table 2.

2.1.1 Plugging in some numbers

Say we want to actually build a four-hole coupler as calculated previously. The coupler shall operate in the Ka band, from 26.5 GHz to 40 GHz, using a WR-28 waveguide, and the coupling shall be −20 dB.

From theory of rectangular waveguides, one can determine the guided wavelength

$$\lambda_g = \frac{c}{f} \cdot \frac{1}{\sqrt{1 - \left(\frac{c}{2\,a\,f}\right)^2}} \tag{47}$$

which yields $\lambda_{g,1} = 18.67\,\text{mm}$ at 26.5 GHz and $\lambda_{g,2} = 8.82\,\text{mm}$ at 40 GHz. Figure 5 shows how the guided wavelength depends on the frequency.

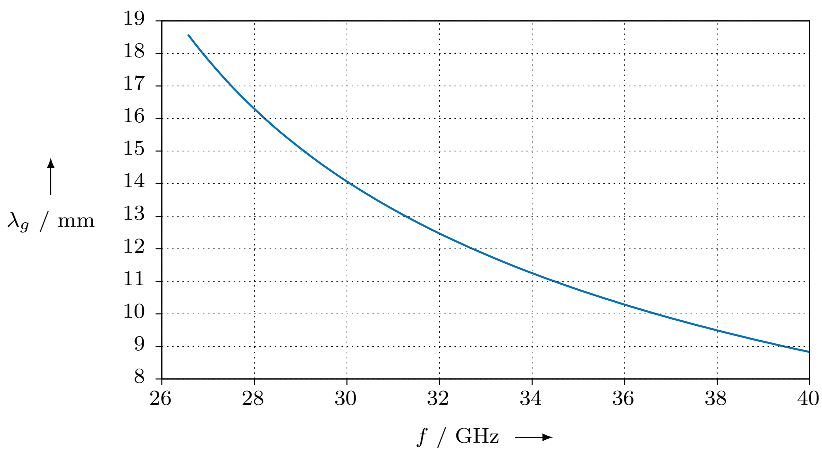


Figure 5: Guided wavelength as function of frequency

The design wavelength of the directional coupler is then the harmonic mean of the two corner wavelengths

$$\lambda_g = \frac{2\,\lambda_{g,1}\,\lambda_{g,2}}{\lambda_{g,1} + \lambda_{g,2}} \tag{48}$$

which is $\lambda_g = 11.98\,\text{mm}$. Since the holes shall have an electrical distance of 90° at the design wavelength, their distance is 2.99 mm. With the physical distance known, it is possible to calculate the electrical distance at each frequency (Equation 20) which is shown in Figure 6. Apparently, the electrical distance ranges from $\phi = 57.7°$ at the lowest frequency to $\phi = 122.3°$ at the highest frequency.

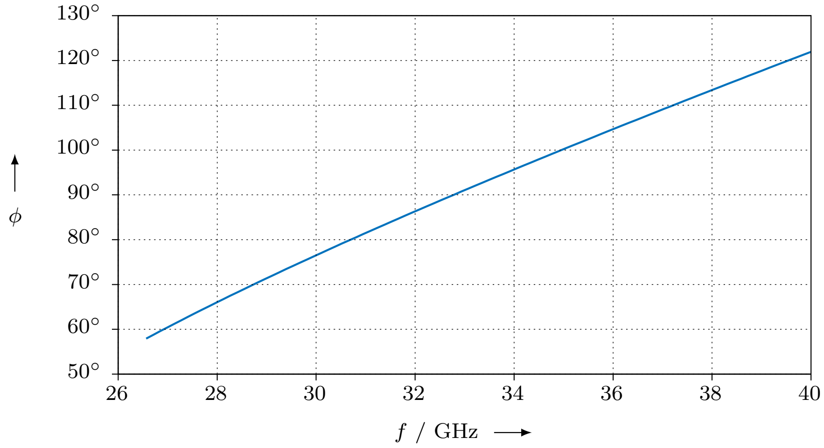


Figure 6: Electrical distance of the holes as function of frequency

The scaling factor t can be found as $t = 1.871$ because $\cos\phi = \pm 0.534$. Then, $t\cos\phi = \pm 1$ at the corner frequencies, which is shown in Figure 7. Taking the harmonic mean to determine the design wavelength, Equation 47, ensures that the hole distance is chosen such that $\cos\phi$ has (besides the sign) the same value at the two corner frequencies.

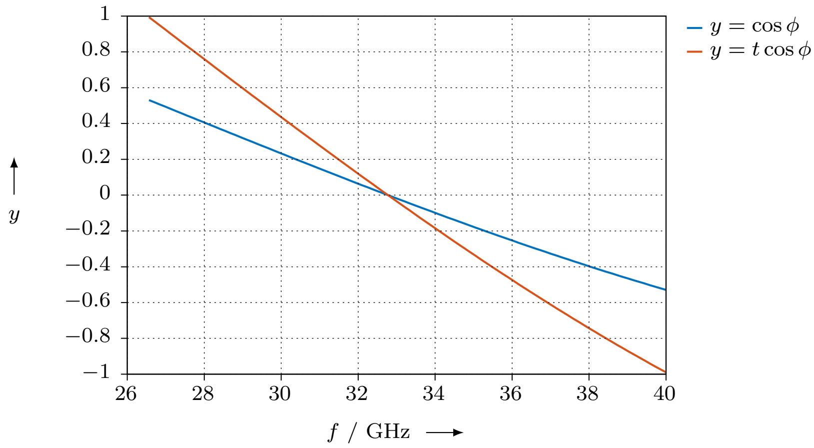


Figure 7: Effect of the scaling factor

Assume now that $a_1 = 1$. Then, according to the previous calculation, $a_2 = 3 - \frac{3}{t^2} = 2.143$. This would result in a theoretical coupling of

$$C = 2\,a_1 + 2\,a_2 = 6.286 \tag{49}$$

but the desired coupling should be 0.1 only for a 20 dB coupler. So we divide a_1 and a_2 by 62.86 and find $a_1 = 0.0159$ and $a_2 = 0.03409$. The theoretical frequency response of this directional coupler is shown in Figure 8. Since the frequency response of a coupler with four holes depends on the Chebyshev polynomial T_3, there are exactly 3 zeros at the frequency response of the isolation.

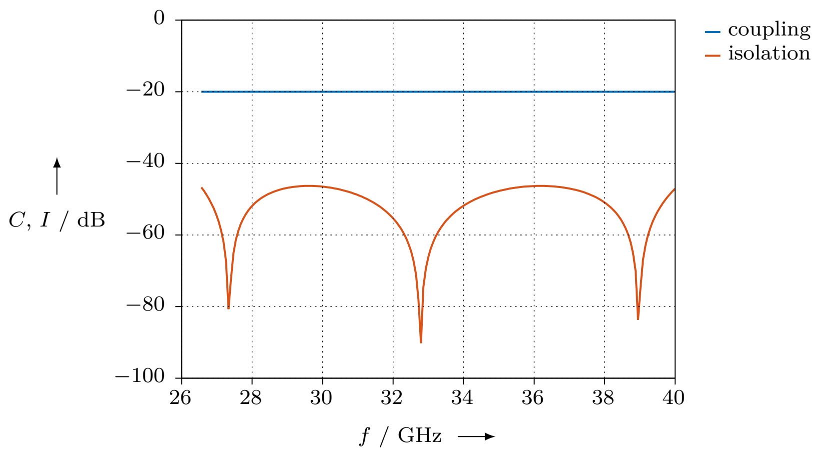


Figure 8: Theoretical frequency response of the four-hole Chebyshev coupler

2.2 Example for five holes

The directional coupler is shown schematically in Figure 9. Because the number of holes is now odd, we have in total 3 different coupling coefficients, a_1, a_2, a_3.

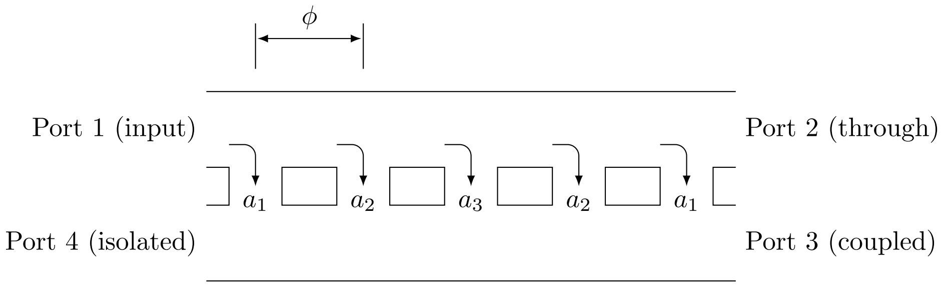


Figure 9: A directional coupler with 5 holes

The coupling coefficient for the isolation is:

$$I = a_1 + a_2\,\mathrm{e}^{-2\,\mathrm{j}\,\phi} + a_3\,\mathrm{e}^{-4\,\mathrm{j}\,\phi} + a_2\,\mathrm{e}^{-6\,\mathrm{j}\,\phi} + a_1\,\mathrm{e}^{-8\,\mathrm{j}\,\phi} \tag{50}$$

Factoring out $\mathrm{e}^{-4\,\mathrm{j}\,\phi}$ yields

$$I = \mathrm{e}^{-4\,\mathrm{j}\,\phi}\left(a_1\mathrm{e}^{4\,\mathrm{j}\,\phi} + a_2\,\mathrm{e}^{2\,\mathrm{j}\,\phi} + a_3 + a_2\,\mathrm{e}^{-2\,\mathrm{j}\,\phi} + a_1\,\mathrm{e}^{-4\,\mathrm{j}\,\phi}\right) \tag{51}$$

which can be written as

$$I = \mathrm{e}^{-4\,\mathrm{j}\,\phi}\,(2\,a_1\cos 4\,\phi + 2\,a_2\cos 2\,\phi + a_3) \quad . \tag{52}$$

The magnitude can be written in terms of Chebyshev polynomials as follows

$$|I| = |2\,a_1\,T_4(x) + 2\,a_2\,T_2(x) + a_3\,T_0(x)| \tag{53}$$

which the reader may easily verify. If we insert the actual Chebyshev polynomials T_4, T_2 and T_0, we find

$$|I| = \left|2\,a_1\,\left(8\,x^4 - 8\,x^2 + 1\right) + 2\,a_2\,\left(2\,x^2 - 1\right) + a_3\right| \tag{54}$$

and we need to satisfy the condition

$$|I| = |T_4(t\,x)| \quad . \tag{55}$$

From here, we find the equation

$$2\,a_1\,\left(8\,x^4 - 8\,x^2 + 1\right) + 2\,a_2\,\left(2\,x^2 - 1\right) + a_3 = T_4(t\,x) \tag{56}$$

or

$$2\,a_1\,\left(8\,x^4-8\,x^2+1\right)+2\,a_2\,\left(2\,x^2-1\right)+a_3=\left(8\,t^4\,x^4-8\,x^2\,t^2+1\right) \tag{57}$$

respectively. This leads to the equation system

$$16\,a_1\,x^4=8\,t^4\,x^4 \tag{58}$$

$$(-16\,a_1+4\,a_2)\,x^2=-8\,t^2\,x^2 \tag{59}$$

$$2\,a_1-2\,a_2+a_3=1 \tag{60}$$

which can be written in matrix notation as follows:

$$\begin{pmatrix}16 & 0 & 0\\ -16 & 4 & 0\\ 2 & -2 & 1\end{pmatrix}\cdot\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}=\begin{pmatrix}8 & 0 & 0\\ 0 & -8 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{61}$$

The solution of this system is:

$$a_1=\frac{1}{2}t^4 \tag{62}$$

$$a_2=2\,t^4-2\,t^2 \tag{63}$$

$$a_3=3\,t^4-4\,t^2+1 \tag{64}$$

Now we can normalise the coupling coefficients with respect to a_1. This yields

$$\frac{a_2}{a_1}=\frac{2\,t^4-2\,t^2}{\frac{1}{2}\,t^4}=\frac{4\,t^4-4\,t^2}{t^4}=4\left(1-\frac{1}{t^2}\right)=4-\frac{4}{t^2} \tag{65}$$

and

$$\frac{a_3}{a_1}=\frac{3\,t^4-4\,t^2+1}{\frac{1}{2}\,t^4}=\frac{6\,t^4-8\,t^2+2}{t^4}=6-\frac{8}{t^2}+\frac{2}{t^4}\quad. \tag{66}$$

Using the same frequency range as in the previous example and again normalising the coupling coefficients to achieve a 20 dB coupling, the frequency response thus obtained is shown in Figure 10.

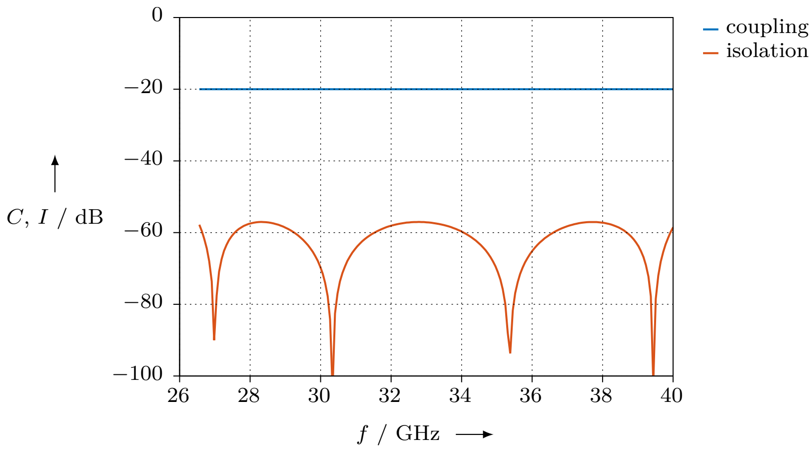


Figure 10: Frequency response of the 5-hole directional coupler

It is easily verified that the polynomials obtained in Equation 65 and Equation 66 are exactly the same ones as shown in [2]. However, the advantage of this method is that it is valid for an arbitrarily large number of holes and does not rely on a table. The full table, published in [2], is also shown in Table 2.

Table 2: Coupler polynomials

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
3	1	$2-\frac{2}{t^2}$	1					
4	1	$3-\frac{3}{t^2}$	$3-\frac{3}{t^2}$	1				
5	1	$4-\frac{4}{t^2}$	$6-\frac{8}{t^2}+\frac{2}{t^4}$	$4-\frac{4}{t^2}$	1			
6	1	$5-\frac{5}{t^2}$	$10-\frac{15}{t^2}+\frac{5}{t^4}$	$10-\frac{15}{t^2}+\frac{5}{t^4}$	$5-\frac{5}{t^2}$	1		
7	1	$6-\frac{6}{t^2}$	$15-\frac{24}{t^2}+\frac{9}{t^4}$	$20-\frac{36}{t^2}+\frac{18}{t^4}-\frac{2}{t^6}$	$15-\frac{24}{t^2}+\frac{9}{t^4}$	$6-\frac{6}{t^2}$	1	
8	1	$7-\frac{7}{t^2}$	$21-\frac{35}{t^2}+\frac{14}{t^4}$	$35-\frac{70}{t^2}+\frac{42}{t^4}-\frac{7}{t^6}$	$35-\frac{70}{t^2}+\frac{42}{t^4}-\frac{7}{t^6}$	$21-\frac{35}{t^2}+\frac{14}{t^4}$	$7-\frac{7}{t^2}$	1

A coupler can be designed only with the aid of this table; the general procedure is as follows:

1. Determine the operating frequency range and the design wavelength.
2. Calculate the electrical hole distance $\phi = \frac{\lambda_g}{4}$.
3. Choose the desired number of holes n.
4. With $t = \cos\phi$, Table 2 yields the coupling coefficients for each hole.

However, using the table can be cumbersome and is not so easy to automate. Therefore, an algorithm is developed to automatically calculate these coupler polynomials, such that arbitrarily large couplers can be designed.

2.3 Algorithm to calculate the coupling coefficients

Listing 2 shows a simple algorithm which calculates the coupling coefficients for an arbitrary number of holes.

The results of the algorithm can be interpreted as follows. Assume `couplerpoly(5)` is called to determine the coupling coefficients for a 5-hole coupler. The result returned would be the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 & 4 \\ 2 & 0 & -8 & 0 & 6 \\ 0 & 0 & -4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which can be interpreted as follows

$$\begin{aligned} a_1 &= 1 \\ a_2 &= -\frac{4}{t^2} + 4 \\ a_3 &= \frac{2}{t^4} - \frac{8}{t^2} + 6 \\ a_4 &= -\frac{4}{t^2} + 4 \\ a_5 &= 1 \end{aligned}$$

and corresponds to Table 2. Thanks to the algorithm, one can calculate the coupler polynomials for an arbitrarily large number of holes.

2.4 From coupling coefficients to hole diameters

So far we have considered the calculation of the coupling coefficients to achieve a desired frequency response of the coupler. But how to find the required hole diameter?

According to Bethe's small hole coupling theory [4], when $x = \frac{a}{4}$, the coupling of a circular hole is

$$|C| = \frac{\pi\, d^3}{12\, a^2\, b} \tag{67}$$

so, to achieve a desired coupling, the hole diameter can be found by

$$d = \sqrt[3]{\frac{12\, a^2\, b}{\pi}\, |C|} \quad . \tag{68}$$

However, this calculation does not take into account that the hole's distance from the wall, x, may be different than $\frac{a}{4}$, and the finite metal thickness and hole size corrections mentioned previously are also not taken into account. Therefore, Equation 68 gives only a initial guess. To find the optimal hole diameters, an iterative procedure can be used, as depicted in Figure 11. This algorithm calculates the actually achieved coupling coefficient based on the hole diameters, taking the various corrections into account (Section 1.3). If the coupling of one particular hole is too low or too high, the hole diameter is adjusted, until all holes have a diameter such that their resulting coupling coefficients are within a specified tolerance.

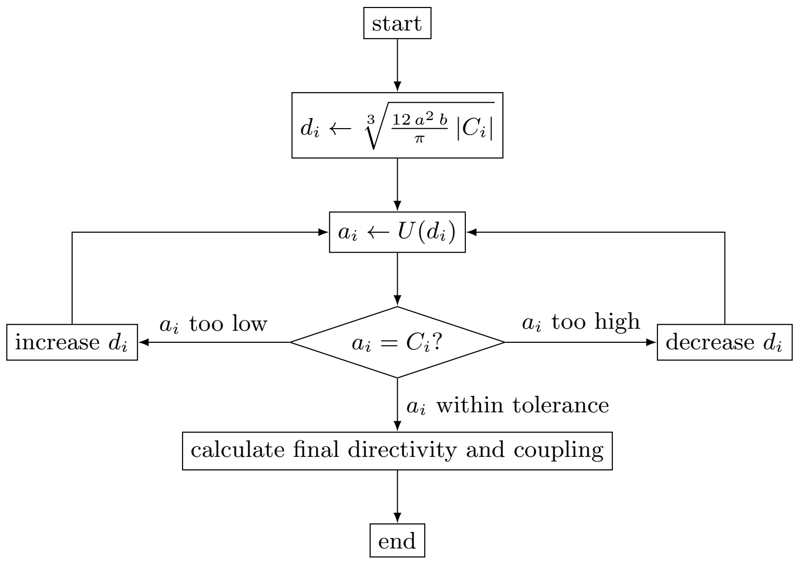


Figure 11: Flowchart to optimise the hole diameters

Of course, sophisticated optimisation algorithms like gradient optimisation or Nelder-Mead could be applied to this problem; however, this "brute-force" optimisation has been used with success because the available computing power is usually large with today's PCs and the number of holes is so small that the additional complexity of a sophisticated optimisation algorithm is usually of small benefit. Anyway, there will be manufacturing tolerances in the end, therefore it makes no sense to calculate the hole diameters to a large amount of decimal places (depending on the manufacturing technique).

3 Superimposed arrays and multiple rows of holes

With the theory and algorithms described so far, it becomes possible to calculate the diameters for an arbitrary number of holes, to achieve a given coupling. However, there may still arise some problems.

Consider a directional coupler having 24 holes at $x = 0.203$. The smallest hole would have a diameter of 0.283 mm, while the largest one would be 2.387 mm. It is obvious that the span between the smallest and the largest hole is very large, which is more difficult to manufacture. Especially the small holes are difficult and thus more expensive.

Therefore, superimposed arrays can be used. For instance, 3 superimposed arrays of 8 holes can be viewed as follows:

a_1	a_2	a_3	a_4	a_4	a_3	a_2	a_1								
				a_1	a_2	a_3	a_4	a_4	a_3	a_2	a_1				
								a_1	a_2	a_3	a_4	a_4	a_3	a_2	a_1
a_1	a_2	a_3	a_4	$a_4 + a_1$	$a_3 + a_2$	$a_2 + a_3$	$a_1 + a_4$	$a_4 + a_1$	$a_3 + a_2$	$a_2 + a_3$	$a_1 + a_4$	a_4	a_3	a_2	a_1

Instead of physically having 3 rows of holes, one can add the coupling coefficients, as shown in the last row. If the number of holes is odd, a possible arrangement is shown below:

a_1	a_2	a_3	a_4	a_3	a_2	a_1						
			a_1	a_2	a_3	a_4	a_3	a_2	a_1			
						a_1	a_2	a_3	a_4	a_3	a_2	a_1
a_1	a_2	a_3	$a_4 + a_1$	$a_3 + a_2$	$a_2 + a_3$	$a_1 + a_4 + a_1$	$a_3 + a_2$	$a_2 + a_3$	$a_1 + a_4$	a_3	a_2	a_1

The advantage of these superimposed arrays is that the variation of the hole diameters is much smaller. Figure 12 shows the diameter of each hole for a coupler having 24 holes, for different array configurations. The 24-1 array is one array of 24 holes, whereas the 8-5 array are 5 superimposed arrays of 8 holes each, and the 4-11 array is 11 arrays of 4 holes each. As one can see, the less holes there are, the smaller is the change from the smallest to the largest hole.

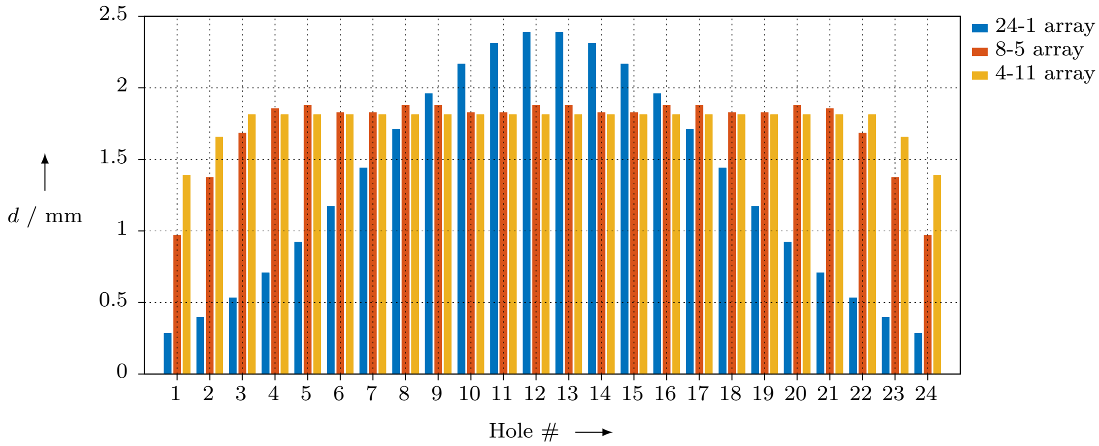


Figure 12: Comparison of the hole diameters for differrent array configurations

4 Simulation of couplers

4.1 Effect of manufacturing tolerances

The most sensitive dimension seems to be the diameter of the holes. The coupling coefficient is not very sensitive to the hole diameter, but for a good coupler, one wants the isolation to be as high as possible, i.e. the signal on the "isolated" port is very small. If the diameter of one hole changes, the overall coupling will not be affected very much, but the isolation is likely to be affected because the signal on the "isolated" port is so small.

So, a simple Monte-Carlo simulation has been performed for the hole diameters. A coupler having 6 superimposed arrays of 12 holes was designed. The coupling was 10 dB. Then, the hole diameters have been changed randomly by ±0.05 mm. 400 runs have been done and the frequency response was plotted each time. Figure 13 shows the result with the different curves overlayed.

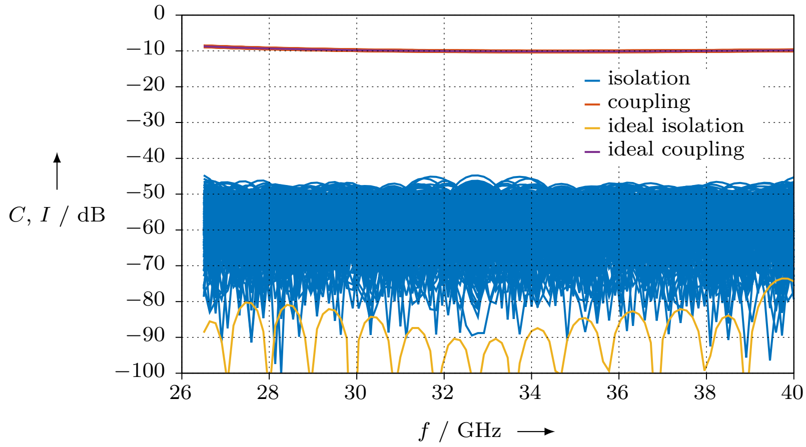


Figure 13: Comparison of the hole diameters for differrent array configurations

For reference, the “ideal” isolation and coupling are shown as well, which means in this case that the hole diameters are all rounded to 0.05 mm but without tolerances. As one can see, as soon as the tolerances are introduced, the coupling barely changes while the isolation is severely affected and thus leads to bad directivity values.

4.2 Simulation in Matlab vs. full-wave

The simple simulation in Matlab does only provide approximate values for the coupling and the isolation. There are no values available for the return loss and the insertion loss. Figure 14 shows a comparison of the simulation with a full-wave simulator. As one can see, the results are quite similar. The main difference comes from the fact that the Matlab simulation does not take into account the interaction of the coupling holes, while the full-wave simulation does.

However the Matlab simulation is still advantageous because it provides the coupling value which is very close to the real one and it runs much faster than any full-wave simulation using, e.g. FEM or FTDT techniques.

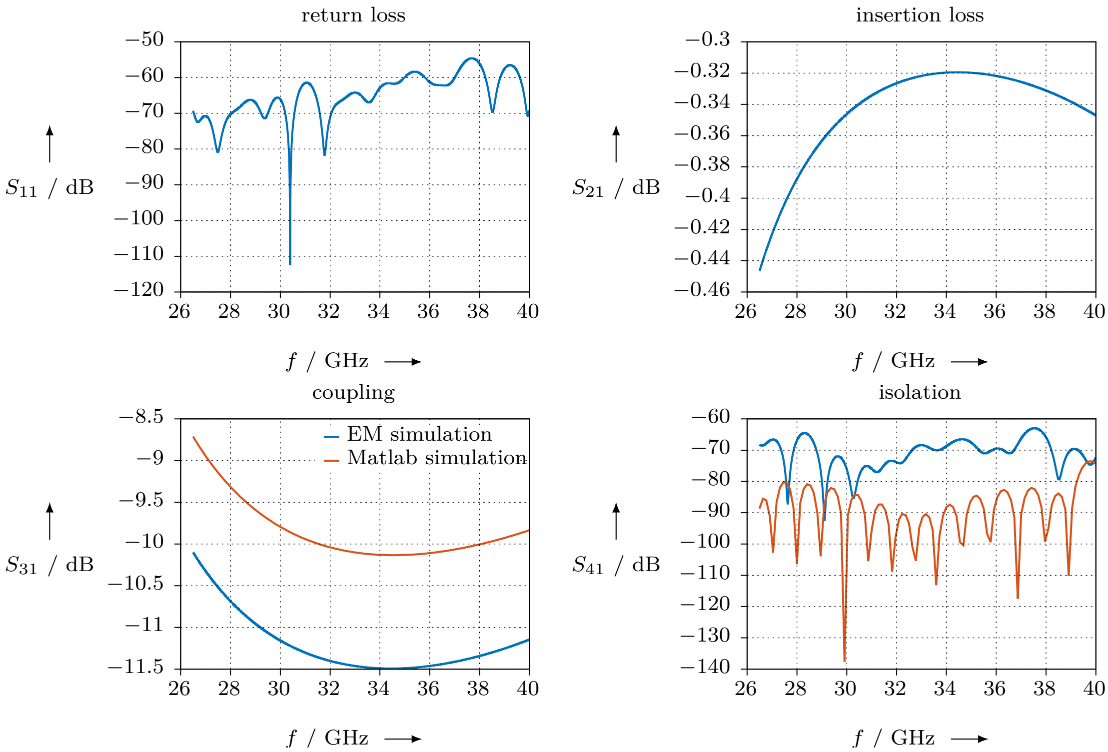


Figure 14: Simulation of a coupler having 12 holes in 6 arrays in 2 rows

5 Construction of a directional coupler

5.1 First attempt

A directional coupler with the following specifications shall be manufactured:

- coupling −10 dB
- WR-28 waveguide ($a = 7.11$ mm, $b = 3.55$ mm)
- frequency range 26.5 GHz to 40 GHz – Ka-band
- no internal termination for the isolated port, i.e. it is a bidirectional coupler
- 2 rows of coupling holes
- 5 superimposed Chebyshev arrays of 8 holes each

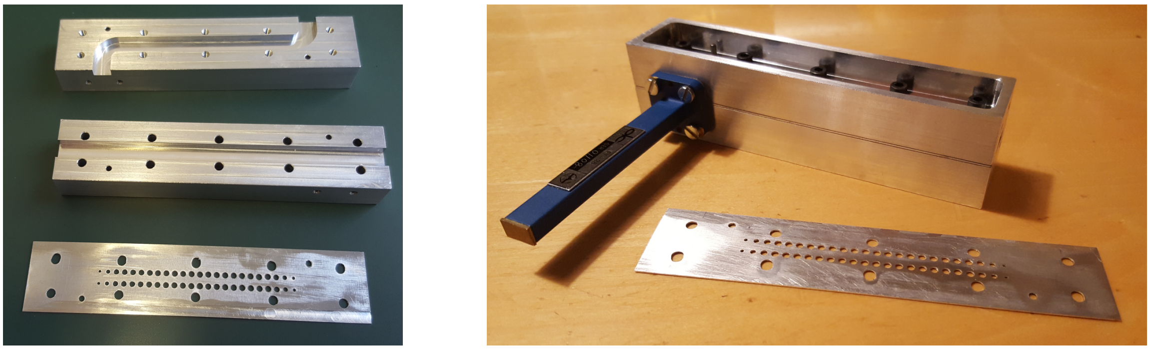
The Matlab script was used to calculate the dimensions and the spacing of the coupling holes. The hole diameters according to Table 3 were found, while the spacing between the holes was found to be 2.99 mm (rounded to 3 mm) and the spacing of the two rows was determined to be 4.22 mm.

Table 3: Diameters of the coupling holes

	hole diameters / mm	
#	calculated	rounded
1	1.0481	1.0
2	1.4821	1.5
3	1.8147	1.8
4	1.9950	2.0
5	2.0197	2.0
6	1.9655	2.0
7	1.9655	2.0
8	2.0197	2.0
9	2.0197	2.0
10	1.9655	2.0
11	1.9655	2.0
12	2.0197	2.0
13	2.0197	2.0
14	1.9655	2.0
15	1.9655	2.0
16	2.0197	2.0
17	2.0197	2.0
18	1.9655	2.0
19	1.9655	2.0
20	2.0197	2.0
21	1.9950	2.0
22	1.8147	1.8
23	1.4821	1.5
24	1.0481	1.0

Figure 15a shows the three main parts of the directional coupler. The top and bottom pars of the waveguide have been CNC milled from 6061 aluminium. The perforated sheet bas been made by gluing a piece of 0.5 mm thick aluminium sheet onto a base plate. The holes were then drilled manually and the contour was also manually milled. Afterwards, the perforated sheet could be removed from the base plate by heating it with a hot air gun, thus melting the glue.

The parts are mounted together by means of 10 M3 screws. Precise alignment is ensured by two dowel pins. Figure 15b shows the assembled directional coupler with an externally mounted termination.



(a) parts (b) assembled

Figure 15: Manufactured waveguide directional coupler

After assembly, the directional coupler's performance was measured using a Rohde & Schwarz ZVA40 network analyser. Figure 16 shows the measured performance of the coupler, together with a full-wave simulation using HFSS.

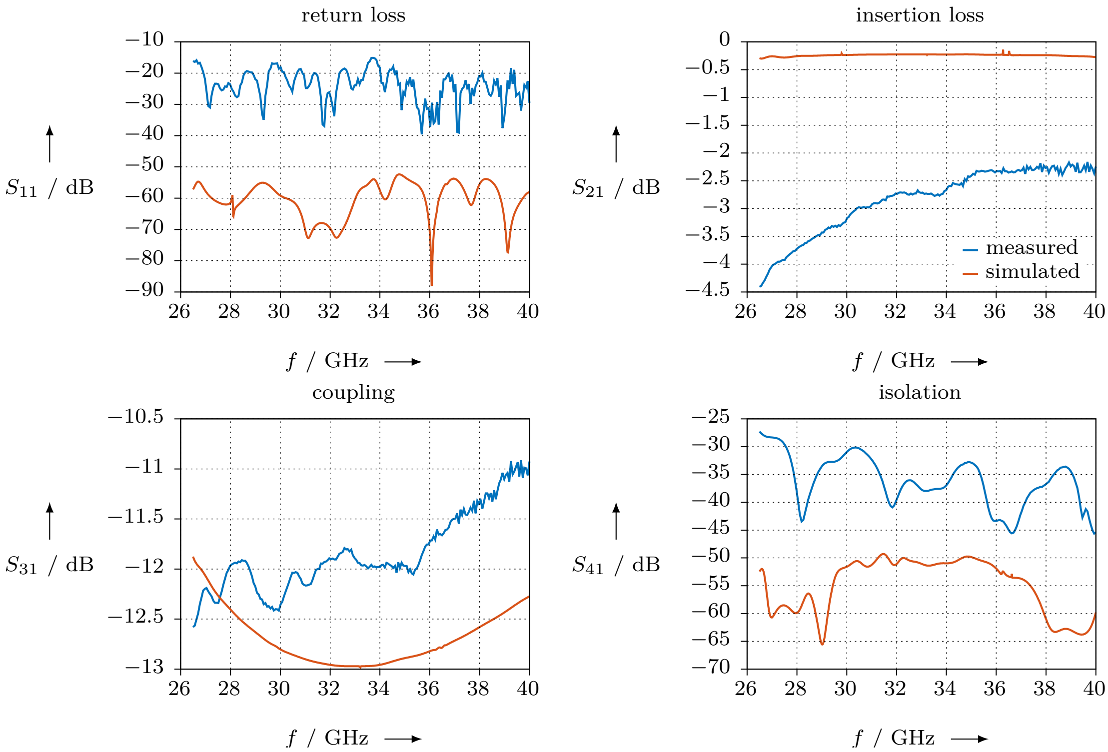


Figure 16: Measured performance of the first version of the coupler

One can see from Figure 16 that the insertion loss is quite high, up to approx. 4.5 dB. Further, the coupling is 1 dB to 2 dB too low. The isolation is only 30 dB in the worst case, giving roughly a directivity of 20 dB. The return loss is around 15 dB to 30 dB. The poor directivity could be either related to the same effect causing the high insertion loss, but it may also be due to the tolerances of the hole diameters.

However, it should be noted that the waveguide adapters required to connect the ZVA40 outputs to the DUT have not been calibrated out, so they will indeed affect both, the return loss and the insertion loss measurements. In fact, the return loss measured is so high that it is likely that it is actually dominated by the waveguide adapters.

To improve the insertion loss, a small recess was milled on the bottom part of the waveguide. Without the recess, the pressure originating from the screws is distributed over a relatively large surface area, giving a low pressure. With the recess shown in Figure 17, the pressure along the waveguide walls is higher, such that a good electrical contact is ensured.

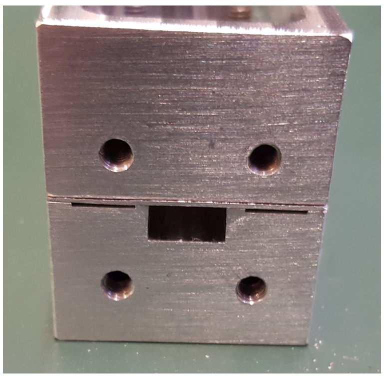


Figure 17: Recess to increase the pressure on the waveguide walls

Indeed, having the recess dramatically improves the performance of the device. Figure 20 shows the comparison of the measured performance of the device before and after making the recess. The return loss has not improved much. Again, the waveguide to coaxial adapters have not been calibrated out. Nevertheless, the insertion loss has improved quite a bit and is now almost constant around 1.5 dB. Assuming an insertion loss of 0.4 dB for each of the waveguide to coax adapters[2], this yields an insertion loss for the coupler of around 0.7 dB. The coupling has improved as well. With the recess, the coupling variation is around 1 dB, the worst case coupling being 11 dB. Also the isolation has improved. The directivity is now at least 25 dB up to 30 dB, but the improvement is not as dramatic as for the insertion loss. This may be an indicator that the hole diameters are too far off.

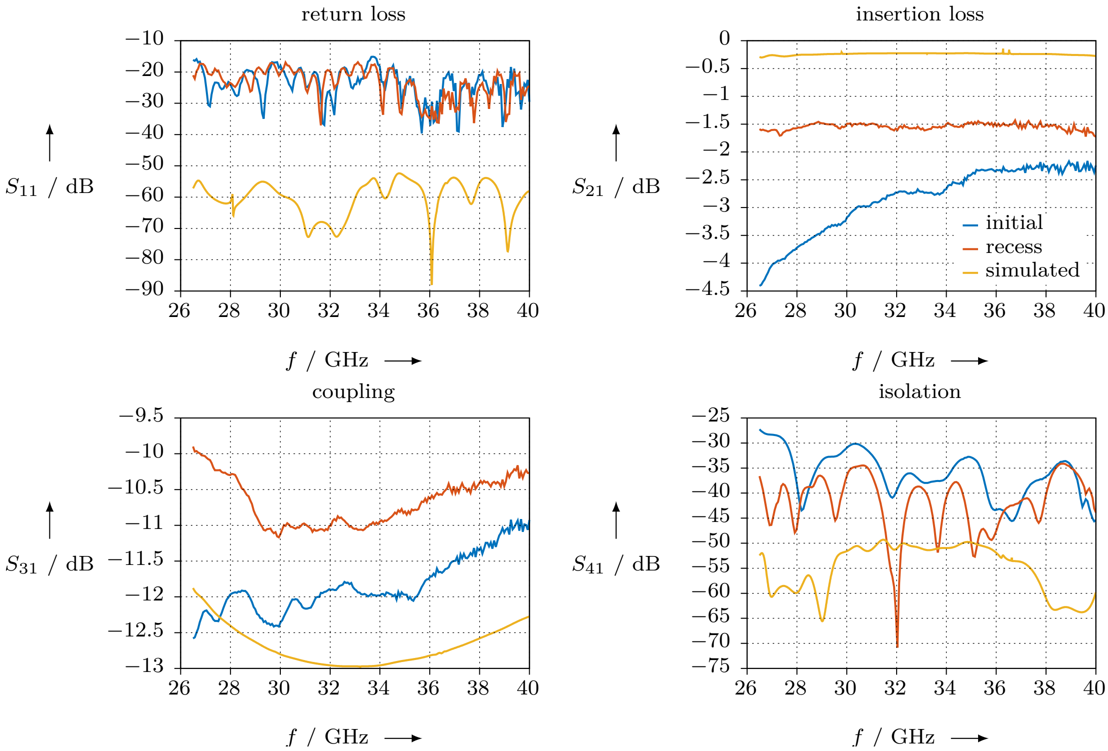


Figure 18: Comparison of the coupler without recess (a) and with recess (b)

It can also be observed that, while all parameters have improved, the return loss stays approximately the same. This is again an indicator that the return loss measured is not that one of the coupler, but just that of the waveguide adapters.

The coupler has been gold-plated after these measurements. The finished coupler is shown in Figure 19.

[2]typical value taken from an adapter from Pasternack

Figure 19: Finished WR-28 directional coupler, 10 dB for Ka-band

After gold-plating, the coupling was measured again and is shown in Figure 20. It is now slightly worse than before; one reason could be improper alignment of the perforated sheet or dirt on the mating surfaces. The plating consists of approx. 2 μm Au over Ni; if the gold thickness is too thin, some current will flow in the nickel layer which has a higher resistance than gold, which could also be an explanation of the worse performance of the coupler.

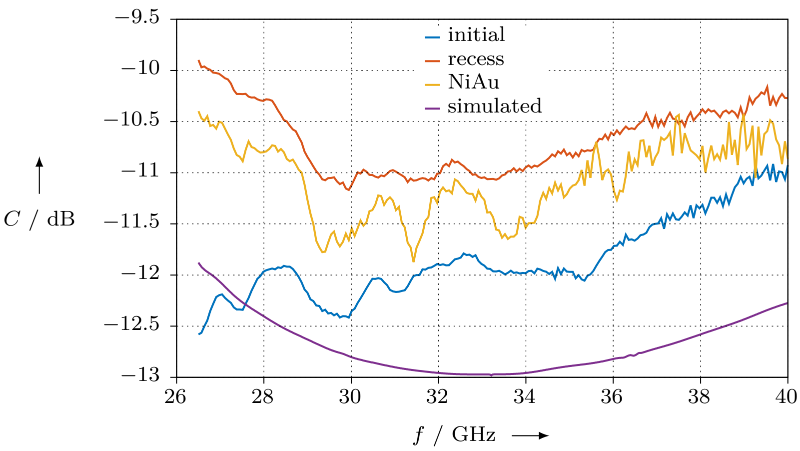


Figure 20: Comparison of the coupler without recess (a) and with recess (b)

It can also be observed that the simulated performance differs significantly from the actually measured performance. After all measurements have been done, it was found that there was a minor mistake in the Matlab script which calculates the hole dimensions and the calculated dimensions were rounded wrong, which leads to a worse coupling coefficient.

A Matlab code

Listing 1: Matlab code to calculate the Chebyshev polynomials

```
function [Tn] = chebypoly(n)
  if n == 0
      Tn = 1;
```

```
        return;
    elseif n == 1
        Tn = [1 0];
        return;
    else
        Tn1 = chebypoly(n-1);
        Tn2 = chebypoly(n-2);

        Tn = 2*conv([1 0], Tn1) - [0 0 Tn2];
        return;
    end
end
```

Listing 2: Matlab code to calculate the coupler polynomials

```
function p = couplerpoly(N)

if mod(N, 2)
    A = zeros((N+1)/2, (N+1)/2);
else
    A = zeros(N/2, N/2);
end

for ind = 1:N/2
    num = N-2*ind+1;
    col = [zeros(1, N-num-1) 2*chebypoly(num)]';
    A(:, ind) = col(1:2:end);
end
if mod(N, 2)
    row = [zeros(1, N-1) chebypoly(0)];
    A(end, :) = A(end, :) + row(1:2:end);
end

p = chebypoly(N-1);
B = diag(p(1:2:end));

result = A \ B;

polys = zeros(size(result, 1), N);
polys(:, 1:2:end) = result(:, :);

numerator = polys(1, :);

halfmat = zeros(size(polys));

for ind = 1:size(polys, 1)
    denominator = [polys(ind, :) zeros(1, size(polys, 2)-1)];

    [q, ~] = deconv(denominator, numerator);

    halfmat(ind, :) = q;
end

if mod(N,2)
    halfrows = floor(N/2);
    result = [halfmat; flipud(halfmat(1:halfrows, :))];
else
    result = [halfmat; flipud(halfmat)];
end

p = fliplr(result);

end
```

Listing 3: Code for automatic design of directional couplers

```
% PRELIMINARY VERSION

clear all; close all; clc


% change values below as needed

% coupling in db
cc = 10;

% lower frequency
fl = 26.5e9;

% upper frequency
fh = 40e9;

% waveguide dimensions
a = 7.11e-3;
b = 3.55e-3;

% wall thickness
t = 0.2e-3;

% number of holes
N = 12;

% number of arrays
NA = 6;

% number of rows
NR = 2;

% distance of the holes to the walls, as fraction of a
k = 0.203;

% wavelengths at the two corner frequencies
c = 2.998e8;
ll = c/fl;
lh = c/fh;

% mean frequency (shouldn't this be the geometric mean?) and wavelength
f = (fl + fh) / 2;
lambda = @(f) c/f;

x = k*a;

% lower and upper guided wavelength
ll = 1/sqrt(1/ll^2 - 1/(2*a)^2);
lh = 1/sqrt(1/lh^2 - 1/(2*a)^2);

lg = @(f) 1/sqrt(1/lambda(f)^2 - 1/(2*a)^2);


ca = 1-c;


% calculate hole spacing.
sp = ll*lh/(2*(ll+lh));
fprintf('hole spacing: %f mm\n', sp*1000);

% electrical distance of the holes
phi = 2*pi*sp/ll;
ww = cos(phi);

aet = @(d) 1.0103*t + 0.0579*d/2;
```

```
amt = @(d) 1.0064*t + 0.0819*d/2;

ke = @(d,f) exp(-2*pi*aet(d)./(1.305*d) .* sqrt(1-(1.305*d/lambda(f)).^2));
km = @(d,f) exp(-2*pi*amt(d)./(1.705*d) .* sqrt(1-(1.705*d/lambda(f)).^2));

qe = @(d,f) 2*lambda(f)./(pi*1.305*d) .* tan(pi*1.305*d./(2*lambda(f)));
qm = @(d,f) 2*lambda(f)./(pi*1.705*d) .* tan(pi*1.705*d./(2*lambda(f)));

u1 = @(d,f) sin(pi*x/a)^2/6 .* km(d,f) .* qm(d,f);
u2 = @(d,f) cos(pi*x/a)^2*lg(f)^2/(24*a^2) .* km(d,f) .* qm(d,f);
u3 = @(d,f) -sin(pi*x/a)^2*lg(f)^2/(12*lambda(f)^2) .* ke(d,f) .* qe(d,f);
U = @(d,f) 2*pi*d.^3/(a*b*lg(f)) .* (u1(d,f) + u2(d,f) + u3(d,f));



% find the coupler polynomial.
poly = couplerpoly(N);
A = zeros(size(poly,1), 1);
for row = 1:size(poly,1)
    A(row) = polyval(poly(row,:), ww);
end



Aa = 20*log10(A / sum(A));

N2 = ceil(N/2);

% total number of holes per row
NH = N + (NA-1)*N2;
fprintf('number of holes per row: %d\n', NH);

AS = zeros(NA, NH);

for j=1:NA
    for k=1+(j-1)*N2 : NH
        ind = k-(j-1)*N2;
        if ind > length(A)
            AS(j,k) = 0;
        else
            AS(j,k) = A(k-(j-1)*N2);
        end
    end
end

s = sum(sum(AS));

if NA > 1
    co = 20*log10(s ./ sum(AS)) + cc + 6.0206*(NR-1);
else
    co = 20*log10(sum(AS) ./ AS) + cc + 6.0206*(NR-1);
end

d_initial = nthroot(12*a^2*b./(pi*10.^(co/20)),3);

func = @(d) norm(20*log10(1./U(d,f)) - co);
d_best = fminsearch(func, d_initial, optimset('MaxFunEvals', 30000, 'MaxIter', 30000));

% best and rounded hole diam.
d = round(50e3*d_best)/50e3;
fprintf('initial, best and rounded hole diameters:\n');
disp([d_initial' d_best' d'])

cnew = 20*log10(1./U(d,f));

% actual coupling
```

```matlab
coup = 20*log10(sum(10.^(-cnew/20)))+6.0206*(NR-1);
fprintf('actual coupling: %f\n', coup);
fprintf('distance of the rows: %f mm\n', (a-2*x)*1e3)
fprintf('total length of coupled section: %f\n', NH*sp*1e3)

% simulation
f = linspace(fl, fh);

k = zeros(length(f), 4);


dist = (0:NH-1)*sp;

for ind = 1:length(f)



phase1 = -2*pi/lg(f(ind))*dist;
isol_ideal = 20*log10(abs(sum(U(d_best, f(ind)) .* exp(-1i*2*phase1))));
isol = 20*log10(abs(sum(U(d, f(ind)) .* exp(-1i*2*phase1))));

isol_max = 20*log10(abs(sum(U(d+0.02e-3, f(ind)) .* exp(-1i*2*phase1))));
isol_min = 20*log10(abs(sum(U(d-0.02e-3, f(ind)) .* exp(-1i*2*phase1))));

phase2 = fliplr(phase1);
phase = phase1 + phase2;
num = (0:NH-1);
koppel_ideal = 20*log10(abs(sum(NR*(U(d_best, f(ind)) .* exp(-2i *phi)))));
koppel = 20*log10(abs(sum(NR*(U(d, f(ind)) .* exp(-2i *phi)))));

koppel_max = 20*log10(abs(sum(NR*(U(d+0.02e-3, f(ind)) .* exp(-2i *phi)))));
koppel_min = 20*log10(abs(sum(NR*(U(d-0.02e-3, f(ind)) .* exp(-2i *phi)))));



k(ind, :) = [isol_ideal isol koppel_ideal koppel];


end

figure(1);
plot(f/1e9, k);
grid on
legend('isolation ideal', 'isolation rounded', 'coupling ideal', 'coupling rounded')
xlabel('f / GHz');
ylabel('dB');



montecarlo = 400;

xplus = 0.05e-3;
xminus = -0.05e-3;

r = [0 1; 1 1] \ [xplus; xminus];

res_isol = zeros(montecarlo, length(f));
res_koppel = res_isol;

for run = 1:montecarlo

    tol = r(1)*rand(size(d)) + r(2);



    for ind = 1:length(f)
```

```
        phase1 = -2*pi/lg(f(ind))*dist;
        isol = 20*log10(abs(sum(U(d+tol, f(ind)) .* exp(-1i*2*phase1))));

        phase2 = fliplr(phase1);
        phase = phase1 + phase2;
        num = (0:NH-1);

        koppel = 20*log10(abs(sum(NR*(U(d+tol, f(ind)) .* exp(-2i *phi)))));



        res_isol(run, ind) = isol;
        res_koppel(run, ind) = koppel;

    end



end

figure(2);

plot(f, res_isol, 'r', f, res_koppel, 'b');


if 1
op = fopen('output.txt', 'w');

for ind = 1:montecarlo


    for j = 1:length(f)

        fprintf(op, '%f %f %f\n', f(j), res_isol(ind, j), res_koppel(ind, j));
    end
    fprintf(op, '\n\n');


end

fclose(op);

end
```

References

[1] R. Levy, *A guide to the practical application of Chebyshev functions to the design of microwave components*, Institution of Engineering and Technology (IET), vol. 106, no. 10, pp. 193–199, 1959.

[2] L. Young, *Advances in Microwaves*, vol. 1, Academic Press, 1966

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[4] H. A. Bethe, *Theory of Diffraction by Small Holes*, Phys. Rev., vol. 66, pp. 163–182, 1944.